Stochastic perturbation-based finite element approach to fluid flow problems

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Abstract

Purpose – To generalize the traditional 2nd order stochastic perturbation technique for input random variables and fields and to demonstrate for flow problems.

Design/methodology/approach – The methodology is based on an *n*-th order expansion (perturbation) for input random parameters and state functions around their expected value to recover probabilistic moments of the response. A finite element formulation permits stochastic simulations on irregular meshes for practical applications.

Findings – The methodology permits approximation of expected values and covariances of quantities such as the fluid pressure and flow velocity using both symbolic and discrete FEM computations. It is applied to inviscid irrotational flow, Poiseulle flow and viscous Couette flow with randomly perturbed boundary conditions, channel height and fluid viscosity to illustrate the scheme.

Research limitations/implications – The focus of the present work is on the basic concepts as a foundation for extension to engineering applications. The formulation for the viscous incompressible problem can be implemented by extending a 3D viscous primitive variable finite element code as outlined in the paper. For the case where the physical parameters are temperature dependent this will necessitate solution of highly non-linear stochastic differential equations.

Practical implications – Techniques presented here provide an efficient approach for numerical analyses of heat transfer and fluid flow problems, where input design parameters and/or physical quantities may have small random fluctuations. Such an analysis provides a basis for stochastic computational reliability analysis.

Originality/value – The mathematical formulation and computational implementation of the generalized perturbation-based stochastic finite element method (SFEM) is the main contribution of the paper.

Keywords Flow, Fluids, Finite element analysis, Stochastic processes

Paper type Research paper

1. Introduction

Perturbation techniques have been extensively used in applied mathematics and particularly for problems in fluid mechanics. In fact, the ideas of boundary-layer behavior for singular perturbation theory and matched asymptotics have their origin

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in the pioneering work of Prandtl (1904) for flow in a viscous boundary-layer adjacent to a wall. In this case, scaling and inspectional analysis led to a significant model reduction from a full viscous flow equation system to the simpler boundary-layer equations. Prandtl's boundary-layer equations have subsequently been derived and analyzed more formally using singular perturbation theory and matched asymptotics.

Regular perturbation techniques are simpler than singular perturbation methods, and also have been utilized to advantage in model simplification. For instance, Lighthill (1954) studied potential flow in the region above a "wavy wall". The shape of the wall in Lighthill's example corresponded to a sinusoidal wave of very small amplitude ε relative to the wavelength. Expanding the potential $\phi(x, y, \varepsilon)$ in terms of the wall amplitude parameter, yields a sequence of Poisson problems for the respective expansion potentials, all posed on the upper half-plane with "flat" boundary $y = 0$ and admitting successive analytic solutions. In this case, the regular perturbation expansion leads to domain simplification rather than simplification of the governing equation. Similar ideas have been exploited in thin airfoil theory. In an even earlier example, Rayleigh (1916) developed a regular perturbation expansion technique for subsonic 2D potential flow. Here, conservation of mass together with the Bernoulli relation lead to the non-linear full potential equation, and the problem can be recast after scaling so that the incident Mach number M_∞ enters explicitly as a small parameter $\varepsilon = M_\infty^2$ in the non-linear equation. A perturbation expansion of the potential or stream function solution was utilized to obtain a 1st order compressibility correction in the form of an analytic solution for compressible flow past a cylinder in an infinite stream.

Of course, analytic solutions are possible only for relatively simple domains and governing equations but the perturbation approach can also be used in conjunction with numerical methods. For example, Carey (1975) developed a combined perturbation and variational formulation for the subsonic potential flow problem class studied by Rayleigh. The objective of the present work is an analogous extension of this idea in the following sense: we combine the concepts of regular perturbation techniques and variational finite element methods with stochastic moment approaches to treat uncertainty in inflow or wall boundary conditions, in parameters of the equations, in profile shape and so on.

Stochastic effects connected with random variations in data are becoming increasingly important considerations in a computer simulation for engineering analysis and design. Techniques are being implemented to determine results in the form of mean values and standard deviations of solution variables as well as physical parameters determined from the simulations. These ideas are more extensively developed in structural dynamics because of the interest in analysis of random vibrations. At present, several different probabilistic schemes have been formulated including, for instance, the stochastic finite element method (SFEM) (Kamiński, 2001a, b, 2003; Kleiber and Hien, 1992), stochastic spectral approaches (Ghanem and Spanos, 1991) and various Monte-Carlo simulation (MCS) techniques (Hurtado and Barbat, 1998).

Moreover, the interest in quantifying errors in simulation has expanded from a posteriori analysis of discretization errors to embrace errors due to modeling approximations and most recently to consider the influence of uncertainty in data. The present work has some bearing on this latter point. The situation is again analogous to that of regular perturbation theory where the 1st order perturbation correction provides an indicator of the modeling error if the simpler 0 order model were chosen.

Likewise in the present context the higher order stochastic perturbation terms play an analogous role and give an indication of the relative impact of uncertainty in the data on the output. Furthermore, one may be interested in a particular result such as the expectation and variance of the output at a point and wish to compare or calibrate the order of errors due to discretization, modeling and uncertainty.

The present paper considers stochastic fluid flow problems in which certain input flow parameters such as material parameters, boundary data or boundary shapes enter as random fields defined by their first two probabilistic moments. In the 2nd order method all state functions in the variational formulation are expanded by use of up to 2nd order Taylor series perturbation expansions with respect to some input random parameters of the flow. As a result, the expected values and cross-covariances of any state function (fluid velocities or temperatures, for instance) may be obtained. This method is, at present, the most efficient choice. However, it imposes limitations on the coefficients of variation for the input random fields due to the nature of the perturbation assumptions. Because of these limitations, the stochastic convergence of the perturbation method is analyzed using simple symbolic computations to compare the 2nd, 4th and 6th order approximation of the expected values against the deterministic solution for a simple test problem with the random output being inversion of the Gaussian random variable. This study confirms that, as widely discussed in the literature, convergence of the perturbation technique strongly depends on the transformation type between the output and input random variables. We begin with a formal derivation of the stochastic 2nd order moment variational finite element method that is the main goal of this present study and illustrates the ideas for a class of flows of sufficient complexity. In the supporting numerical studies we give several simple illustrative finite element examples corresponding to channel and pipe fully developed flow (pseudo 1D). First we consider Couette flow with zero pressure gradient with random input wall velocity amplitude, where a trivial analytic solution with expectations and variances is easily determined for comparison and validation of the SFEM approach. Then we investigate cases where the random variable is the viscosity and channel width H, respectively, and consider the case with adverse pressure gradient. The analogous problem for similar random input data in pipe flow is also presented. Analytic and SFEM solutions are compared for a representative case. Convergence of the stochastic perturbation scheme is also investigated numerically. Finally, we consider a 2D analytic example corresponding to a potential flow problem analogous to the "wavy wall" problem studied by Lighthill but for a flow past a "cylinder" with random Gaussian shape perturbation.

2. Mathematical model

2.1 Variational formulation for deterministic fluid flow

To demonstrate the basic approach in an application setting of general interest, let us consider viscous incompressible flow of a Newtonian fluid in region Ω. The equations of momentum and continuity as well as Stokes constitutive relation can be written as (White, 1986)

$$\rho\left(\frac{\partial v_i}{\partial t}+v_{i,j}v_j\right)=\sigma_{ij,j}+f_i^{\mathrm{B}}, \tag{1}$$

$$v_{i,i} = 0, \tag{2}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{3}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}). \tag{4}$$

The state variables v_i, ε_{ij}, σ_{ij} denote velocity, strain and stress tensor components, respectively, and f_i^B are the body force components. The variables ρ, p, μ denote mass density, pressure, and viscosity, respectively.

Typical boundary conditions for these equations are:

$$v_i = \hat{v}_i; \quad \mathbf{x} \in \partial\Omega_v \tag{5}$$

for fluid velocities, and

$$\sigma_{ij}n_j = \hat{f}_i; \quad \mathbf{x} \in \partial\Omega_\sigma, \tag{6}$$

for surface tractions, on boundary sub-regions $\partial\Omega_V$ and $\partial\Omega_\sigma$, respectively.

The weak variational formulation of the above problem follows on integration by parts in the corresponding weighted residual integral statement with stress boundary conditions included as natural boundary conditions and velocity boundary conditions being essential conditions. More specifically, multiplying equation (1) by virtual velocity component variation δv_i and applying the Gauss divergence theorem to the integral involving the stress tensor, we have

$$\int_\Omega \delta v_i \rho(\dot{v}_i + v_{i,j}v_j)\mathrm{d}x + \int_\Omega \delta\varepsilon_{ij}\sigma_{ij}\mathrm{d}x = \int_\Omega \delta v_i f_i^B \mathrm{d}x + \int_{\partial\Omega_\sigma} \delta v_i \hat{f}_i \mathrm{d}s \tag{7}$$

where v_i satisfy equation (5) on $\partial\Omega_V$ and $\sigma_{ij}n_j$ has been replaced by $\hat{f}_i$ in the integral on the traction boundary $\partial\Omega_\sigma$. Similarly, introducing a pressure variation δp as a trial function in the weighted integral for equation (2) we have

$$\int_\Omega \delta p v_{i,i}\,\mathrm{d}x = 0 \tag{8}$$

The fluid flow problem is then to solve equations (7) and (8) subject to appropriate initial conditions with σ_{ij} and ε_{ij} given by equations (3) and (4) and specified v_i on $\partial\Omega_V$ for arbitrary admissible test functions δv_i, δp having $\delta v_i = 0$ on $\partial\Omega_V$.

We now introduce the 2nd order perturbation second probabilistic central moment technique (Kamiński, 2001a, b; Kleiber and Hien, 1992) to formulate the corresponding stochastic fluid flow problem.

2.2 Stochastic perturbation approach

For illustrative purposes, let us assume that material parameters ρ, μ as well as boundary conditions may be input Gaussian random fields of the problem and are defined uniquely by their expected values and covariances. (In the supporting numerical studies we also consider a case when the boundary shape is similarly specified using a random field.) Let us denote the corresponding random field vector of the problem by $b_r(x_i)$, with probability density functions (PDF) $p(b_r)$ and $p(b_r, b_s)$, respectively, for $r, s = 1, \ldots, R$, where R represents the total number of different input random fields (Kleiber and Hien, 1992; Vanmarcke, 1983). Then, the first two probabilistic moments of $b_r(x_i)$ are the expectation

$$E[b_r] \equiv b_r^0 = \int_{-\infty}^{+\infty} b_r p(b_r)\,\mathrm{d}b_r, \tag{9}$$

and covariance

$$\mathrm{Cov}(b_r; b_s) \equiv S_{rs} = \int_{-\infty}^{+\infty} \left[b_r - b_r^0\right]\left[b_s - b_s^0\right] p(b_r; b_s)\,\mathrm{d}b_r\,\mathrm{d}b_s. \tag{10}$$

The coefficients of variation of the random input are given by the formula

$$\alpha(b_r) = \frac{\sqrt{\mathrm{Var}(b_r)}}{E[b_r]}. \tag{11}$$

The basic idea of the stochastic perturbation approach follows the classical perturbation expansion idea and is to approximate all input variables and the state functions of the stochastic problem via truncated Taylor series about their spatial expectations in terms of a parameter $\psi > 0$. However, here the small parameter ψ corresponds to a random fluctuation. For example, in the case of random fluid density ρ and viscosity μ, we write the n-th order truncated expression

$$\rho = \rho^0 + \psi\rho^{,r}\Delta b_r + \frac{1}{2}\psi^2\rho^{,rs}\Delta b_r\Delta b_s + \cdots + \frac{1}{n!}\psi^n\rho^{,n}(\Delta b)^n, \tag{12}$$

$$\mu = \mu^0 + \psi\mu^{,r}\Delta b_r + \frac{1}{2}\psi^2\mu^{,rs}\Delta b_r\Delta b_s + \cdots + \frac{1}{n!}\psi^n\mu^{,n}(\Delta b)^n, \tag{13}$$

where

$$\psi\Delta b_r = \psi\left(b_r - b_r^0\right) \tag{14}$$

is the first variation of b_r about its expected value b_r^0

$$\psi^2\Delta b_r\Delta b_s = \psi^2\left(b_r - b_r^0\right)\left(b_s - b_s^0\right) \tag{15}$$

is the second variation of b_r, b_s about b_r^0 and b_s^0, respectively, and the n-th order variation can be expressed as

$$\psi^n(\Delta b)^n = \psi^n\left(b_r - b_r^0\right)^n. \tag{16}$$

The symbol $(\cdot)^0$ represents the value of the function $(\cdot)$ taken at the expectations b_r^0, while $(\cdot)^{,r}$ and $(\cdot)^{,rs}$ denote the first and the second partial derivatives with respect to b_r evaluated at b_r^0 (other problem variables such as boundary conditions and boundary shape may also be subjected to random variations). The state variables v_i and p are similarly expanded to equations (12) and (13)

$$v_i = v_i^0 + \psi v_i^{,r}\Delta b_r + \frac{1}{2}\psi^2 v_i^{,rs}\Delta b_r\Delta b_s + \cdots + \frac{1}{n!}\psi^n v_i^{,n}(\Delta b)^n, \tag{17}$$

$$p = p^0 + \psi p^{,r} \Delta b_r + \frac{1}{2}\psi^2 p^{,rs} \Delta b_r \Delta b_s + \cdots + \frac{1}{n!}\psi^n p^{,n} (\Delta b)^n. \tag{18}$$

Next, substituting the expansions (12)-(13) and (17)-(18) in variational equations (7) and (8) and equating terms of the same order, the 0th, 1st and 2nd orders perturbation equations follow as

- zeroth-order (ψ^0 terms, one coupled partial differential equation system)

$$\begin{aligned} &\int_\Omega \delta v_i \rho^0 \left(\dot{v}_i^0 + v_{i,j}^0 v_j^0 \right) \mathrm{d}x + \int_\Omega \delta\varepsilon_{ij} \left(2\mu^0 \varepsilon_{ij}^0 - p^0 \delta_{ij} \right) \mathrm{d}x \\ &\quad = \int_\Omega \delta v_i \left(f_i^{\mathrm{B}} \right)^0 \mathrm{d}x + \int_{\partial\Omega_\sigma} \delta v_i \left(\hat{f}_i \right)^0 \mathrm{d}s, \end{aligned} \tag{19}$$

$$\int_\Omega \delta p v_{i,i}^0 \mathrm{d}x = 0, \tag{20}$$

- first-order (ψ^1 terms, R coupled partial differential equation system, $r = 1, \ldots, R$) :

$$\begin{aligned} &\int_\Omega \delta v_i \left(\rho^{,r} \dot{v}_i^0 + \rho^0 \dot{v}_i^{,r} + \rho^{,r} v_{i,j}^0 v_j^0 + \rho^0 v_{i,j}^{,r} v_j^0 + \rho^0 v_{i,j}^0 v_j^{,r} \right) \mathrm{d}x \\ &\quad + \int_\Omega \delta\varepsilon_{ij} \left(2\mu^{,r} \varepsilon_{ij}^0 + 2\mu^0 \varepsilon_{ij}^{,r} - p^{,r} \delta_{ij} \right) \mathrm{d}x = \int_\Omega \delta v_i \left(f_i^B \right)^{,r} \mathrm{d}x + \int_{\partial\Omega_\sigma} \delta v_i \left(\hat{f}_i \right)^{,r} \mathrm{d}s, \end{aligned} \tag{21}$$

$$\int_\Omega \delta p v_{i,i}^{,r} \mathrm{d}x = 0, \tag{22}$$

- second-order (ψ^2 terms, one coupled partial differential equation system):

$$\begin{aligned} &\int_\Omega \delta v_i \rho^0 \left(\dot{v}_i^{(2)} + v_{i,j}^{(2)} v_j^0 + v_{i,j}^0 v_j^{(2)} \right) \mathrm{d}x + 2\int_\Omega \delta\varepsilon_{ij} \mu^0 \varepsilon_{ij}^{(2)} \mathrm{d}x \\ &\quad = \left\{ \int_\Omega \delta v_i \left(f_i^B \right)^{,rs} \mathrm{d}x + \int_{\partial\Omega_\sigma} \delta v_i \left(\hat{f}_i \right)^{,rs} \mathrm{d}s \right\} \mathrm{Cov}(b^r, b^s) \\ &\quad - \left\{ \int_\Omega \delta\varepsilon_{ij} \left(4\mu^{,r} \varepsilon_{ij}^{,s} - p^{,rs} \delta_{ij} \right) \mathrm{d}x + 2\int_\Omega \delta v_i \left(\rho^{,r} \dot{v}_i^{,s} + \rho^{,r} v_{i,j}^{,s} v_j^0 \right. \right. \\ &\quad \left. \left. + \rho^0 v_{i,j}^{,r} v_j^{,s} + \rho^{,r} v_{i,j}^0 v_j^{,s} \right) \mathrm{d}x \right\} \mathrm{Cov}(b^r, b^s) \end{aligned} \tag{23}$$

$$\left\{ \int_\Omega \delta p v_{i,i}^{,rs} \mathrm{d}x \right\} \mathrm{Cov}(b^r, b^s) = 0. \tag{24}$$

As in standard regular perturbation solution techniques (Carey, 1975), the 0th order velocity and pressure solution from equations (19) and (20) are needed to obtain the 1st order approximations from equations (21) and (22) and the 2nd order functions from equations (23) and (24) are solved in a similar manner from the known lower order solutions. We can then obtain the first two probabilistic moments of these functions (Kleiber and Hien, 1992; Vanmarcke, 1983) by applying definitions (9) and (10) to any state function $f(b_r;t) \equiv \{p(b_r;t), v_j(b_r;t)\}$. For example, from equation (9) the expectation is

$$\begin{aligned} E[f(t,b_r);b_r] &= \int_{-\infty}^{+\infty} f(t)p(b_r)\mathrm{d}b_r \\ &= \int_{-\infty}^{+\infty} \left(f^0 + \psi f^{,r}\Delta b_r + \frac{1}{2}\psi^2 f^{,rs}\Delta b_r \Delta b_s + \cdots + \frac{1}{n!}\psi^n f^{,n}(\Delta b)^n \right) p(b_r)\mathrm{d}b_r \end{aligned} \tag{25}$$

and the covariance follows similarly from equation (10). If there is high random dispersion in the input random variable, then terms above 2nd order are negligible, so the 2nd order approximation ($n=2$) is adequate and we have

$$\begin{aligned} E[f(t,b_r);b_r] &= \int_{-\infty}^{+\infty} f(t)p(b_r)\mathrm{d}b_r = \int_{-\infty}^{+\infty} \left(f^0 + f^{,r}\Delta b_r + \frac{1}{2} f^{,rs}\Delta b_r \Delta b_s \right) p(b_r)\,\mathrm{d}b_r \\ &= 1 \times f^0(t) + 0 \times f^{,r}(t) + \frac{1}{2} \times f^{,rs}(t) S^{,rs} = f^0(t) + \frac{1}{2} f^{(2)}(t) \end{aligned} \tag{26}$$

where we have set parameter $\psi = 1$ (Kleiber and Hien, 1992) and higher order terms are unnecessary because of high random dispersion of the input random variable. The general n-th order expansion in equation (25) reduces similarly to

$$\begin{aligned} E[f(t,b_r);b_r] &\cong 1 \times f^0(t,b_r) + \frac{1}{2} \times f^{,rs}(t,b_r) \times \mathrm{Cov}(b_r,b_s) \\ &\quad + \frac{1}{4!} \times f^{,rstu}(t,b_r) \times \mathrm{Cov}(b_r,b_s,b_t,b_u) \\ &\quad + \frac{1}{6!} \times f^{,rstuvw}(t,b_r) \times \mathrm{Cov}(b_r,b_s,b_t,b_u,b_v,b_w) \\ &= f^0(t,b_r) + \frac{1}{2} f^{(2)}(t,b_r) + \frac{1}{4!} f^{(4)}(t,b_r) + \frac{1}{6!} f^{(6)}(t,b_r) \end{aligned} \tag{27}$$

where odd order terms are equal to zero for the assumed Gaussian random deviates.

In the case of a single Gaussian input random variable h such as the amplitude of the input velocity, the general expansion simplifies to

$$\begin{aligned} E[f(t,h);h,\psi,m] &= f^0(t,h) + \frac{1}{2}\psi^2 \frac{\partial^2 f}{\partial h^2}\mu_2(h) + \frac{1}{4!}\psi^4 \frac{\partial^4 f}{\partial h^4}\mu_4(h) \\ &\quad + \frac{1}{6!}\psi^6 \frac{\partial^6 f}{\partial h^6}\mu_6(h) + \cdots + \frac{1}{(2m)!}\psi^{2m} \frac{\partial^{2m} f}{\partial h^{2m}}\mu_{2m}(h) \end{aligned} \tag{28}$$

with μ_{2m} denoting the ordinary probabilistic moment of $2m$th order. For Gaussian variables, all central probabilistic moments can be expressed in terms of variances (standard deviations σ) (Vanmarcke, 1983) as

$$\mu_{2k+1}(h) = 0, \mu_{2k}(h) = 1 \times 3 \times \cdots \times (2k-1)\sigma^{2k}(h); \quad k \leq m \tag{29}$$

and can be included easily in computational analysis without any further modifications. For example, for the 6th order expansion this yields

$$\begin{aligned} &\mu_2(h) = \sigma^2(h) = \mathrm{Var}(h), \quad \mu_4(h) = 3\sigma^4(h) = 3\mathrm{Var}^2(h), \\ &\mu_6(h) = 15\sigma^6(h) = 15\mathrm{Var}^3(h). \end{aligned} \tag{30}$$

Using this extension of the random output, a desired efficiency of the expected values can be achieved by the appropriate choice of m and ψ corresponding to the input PDF type, probabilistic moment interrelations, acceptable error of the computations, etc. This choice can be made by comparative studies with MCSs or theoretical results obtained by direct symbolic integration (Cornil and Testud, 2001).

A similar treatment to that above leads to the following result for the cross-correlation of any state function. Recalling equation (10), we have

$$\mathrm{Cov}\left(f\left(x_i^{(1)};t\right), f\left(x_j^{(2)};t\right)\right) = f^{,r}\left(x_i^{(1)};t_1\right) f^{,s}\left(x_j^{(2)};t_2\right) S^{rs} \tag{31}$$

and can be easily extended to higher order formulations. For example, in the 6th order approximation we have

$$\begin{aligned} \mathrm{Cov}(f,g) = &\int_{-\infty}^{+\infty} \Big(f^0 + \Delta b_r f^{,r} + \frac{1}{2}\Delta b_r \Delta b_s f^{,rs} + \frac{1}{3!}\Delta b_r \Delta b_s \Delta b_t f^{,rst} \\ &+ \frac{1}{4!}\Delta b_r \Delta b_s \Delta b_t \Delta b_u f^{,rstu} + \frac{1}{5!}\Delta b_r \Delta b_s \Delta b_t \Delta b_u \Delta b_v f^{,rstuv} - E[f] \Big). \\ &\times \Big(g^0 + \Delta b_c g^{,c} + \frac{1}{2}\Delta b_c \Delta b_d g^{,cd} + \frac{1}{3!}\Delta b_c \Delta b_d \Delta b_e g^{,cde} \Big) \\ &+ \frac{1}{4!}\Delta b_c \Delta b_d \Delta b_e \Delta b_h g^{,cdeh} + \frac{1}{5!}\Delta b_c \Delta b_d \Delta b_e \Delta b_h \Delta b_l g^{,cdehl} - E[g] \Big) p(f(\mathbf{b}), g(\mathbf{b}))\mathrm{d}\mathbf{b} \\ = &\int_{-\infty}^{+\infty} \Big(\Delta b_r f^{,r} + \frac{1}{2}\Delta b_r \Delta b_s f^{,rs} + \frac{1}{3!}\Delta b_r \Delta b_s \Delta b_t f^{,rst} + \frac{1}{4!}\Delta b_r \Delta b_s \Delta b_t \Delta b_u f^{,rstu} \\ &+ \frac{1}{5!}\Delta b_r \Delta b_s \Delta b_t \Delta b_u \Delta b_v f^{,rstuv} \Big) \Big(\Delta b_c g^{,c} + \frac{1}{2}\Delta b_c \Delta b_d g^{,cd} + \frac{1}{3!}\Delta b_c \Delta b_d \Delta b_e g^{,cde} \\ &+ \frac{1}{4!}\Delta b_c \Delta b_d \Delta b_e \Delta b_h g^{,cdeh} + \frac{1}{5!}\Delta b_c \Delta b_d \Delta b_e \Delta b_h \Delta b_l g^{,cdehl} \Big) \times p(f(\mathbf{b}), g(\mathbf{b}))\mathrm{d}\mathbf{b} \end{aligned} \tag{32}$$

Taking into account components resulting in up to 6th order perturbations and eliminating odd order terms (for Gaussian variables), we obtain

$$
\begin{aligned}
\mathrm{Cov}(f,g) = & \int_{-\infty}^{+\infty} \Delta b_r f^{,r} \Delta b_c g^{,c} p(f(b),g(b))\mathrm{d}b \\
& + \int_{-\infty}^{+\infty} \frac{1}{4} \Delta b_r \Delta b_s f^{,rs} \Delta b_c \Delta b_d g^{,cd} p(f(b),g(b))\mathrm{d}b \\
& + \int_{-\infty}^{+\infty} \Delta b_r f^{,r} \frac{1}{3!} \Delta b_c \Delta b_d \Delta b_e g^{,cde} p(f(b),g(b))\mathrm{d}b \\
& + \int_{-\infty}^{+\infty} \Delta b_c g^{,c} \frac{1}{3!} \Delta b_r \Delta b_s \Delta b_t f^{,rst} p(f(b),g(b))\,\mathrm{d}b \\
& + \int_{-\infty}^{+\infty} \frac{1}{3!} \Delta b_r \Delta b_s \Delta b_t f^{,rst} \frac{1}{3!} \Delta b_c \Delta b_d \Delta b_e g^{,cde} p(f(b),g(b))\,\mathrm{d}b \\
& + \int_{-\infty}^{+\infty} \frac{1}{4!} \Delta b_c \Delta b_d \Delta b_e \Delta b_h g^{,cdeh} \frac{1}{2} \Delta b_r \Delta b_s f^{,rs} p(f(b),g(b))\mathrm{d}b \\
& + \int_{-\infty}^{+\infty} \frac{1}{4!} \Delta b_r \Delta b_s \Delta b_t \Delta b_u f^{,rstu} \frac{1}{2} \Delta b_c \Delta b_d g^{,cd} p(f(b),g(b))\mathrm{d}b \\
& + \int_{-\infty}^{+\infty} \frac{1}{5!} \Delta b_r \Delta b_s \Delta b_t \Delta b_u \Delta b_v f^{,rstuv} \Delta b_c g^{,c} p(f(b),g(b))\mathrm{d}b \\
& + \int_{-\infty}^{+\infty} \frac{1}{5!} \Delta b_c \Delta b_d \Delta b_e \Delta b_h \Delta b_l g^{,cdehl} \Delta b_r f^{,r} p\big(f(b),g(b)\big)\mathrm{d}b
\end{aligned} \tag{33}
$$

The first integral above corresponds to the 2nd order perturbation, the next three complete the 4th order approximation and the remainder needs to be included to achieve the full 6th order expansion. Then equation (31) can be written more compactly as

$$
\begin{aligned}
\mathrm{Cov}(f,g) = & \mathrm{Cov}(b_r,b_s) \times f^{,r} g^{,s} + \mathrm{Cov}(b_r,b_s,b_t,b_u)\left(\frac{1}{4} f^{,rs} g^{,tu} + \frac{1}{3!} f^{,r} g^{,stu} + \frac{1}{3!} f^{,rst} g^{,u}\right) \\
& + \mathrm{Cov}(b_r,b_s,b_t,b_u,b_v,b_w) \times \left(\left(\frac{1}{3!}\right)^2 f^{,rst} g^{,uvw} + \frac{1}{4!} g^{,cdeh} \frac{1}{2} f^{,rs} + \frac{1}{4!} f^{,rstu} \frac{1}{2} g^{,cd} \right. \\
& \left. + \frac{1}{5!} f^{,rstuv} g^{,w} + \frac{1}{5!} g^{,rstuv} f^{,w} \right)
\end{aligned} \tag{34}
$$

and, as in the result summarized previously in equation (27), we can provide the formula for a 6th order perturbation variance of a variable v in terms of the Gaussian random input given by U as

$$
\begin{aligned}
\mathrm{Var}(v) = & (v^{,h})^2 \times \mu_2(h) + \left(\frac{1}{4}(v^{,hh})^2 + \frac{2}{3!} v^{,h} v^{,hhh}\right) \times \mu_4(h) \\
& + \left(\left(\frac{1}{3!}\right)^2 (v^{,hhh})^2 + \frac{1}{4!} v^{,hhhh} v^{,hh} + \frac{2}{5!} v^{,hhhhh} v^{,h} \right) \times \mu_6(h)
\end{aligned} \tag{35}
$$

Using equation (30), the 2nd order probabilistic moment of the variable v can be determined as a function of the relevant partial derivatives of v with respect to h as well as using its expected value and variance. Some validation cases with known analytic solution are considered later in subsection 4.1.

Of course, one can compute the desired expectations and variances by Monte Carlo approaches. These involve sampling the random input field and numerous flow solutions for this sample data. In contrast to the stochastic MCS method, the 2nd order perturbation stochastic second moment analysis makes it possible to derive probabilistic moments of the state functions. The use of a 2nd order second moment scheme may be effective and more efficient than Monte Carlo simulation. It should be emphasized, however, that higher order and higher moment approximation is necessary to calculate the output probabilistic moments from the perturbation-based solution. In the next section, we develop a Galerkin semidiscrete finite element formulation of the stochastic 2nd order perturbation equations system (19)-(24) that is then suitable for practical simulations involving small random field parameters.

3. Stochastic finite element method

3.1 Deterministic discretization

For simple operators and domains the stochastic perturbation systems may in some cases admit analytic solutions. We consider such cases in the numerical verification studies later. However, engineering analysis of most problems of practical interest requires discretization by finite elements or a similar scheme. As an example we describe the finite element treatment for the Navier Stokes stochastic perturbation formulation described in the previous section. Introducing a discretization of finite elements with associated velocity and pressure spaces satisfying the inf-sup stability criterion, the velocity and pressure expressions have the form

$$v_i(x_j;t) = \Phi_\alpha(x_j)V_{\alpha i}(t), p(x_j;t) = \tilde{\Phi}_\gamma(x_j)P_\gamma(t), \quad i,j = 1,2,3; \quad \alpha = 1,\ldots,N; \quad \gamma = 1,2,\ldots,M, \tag{36}$$

where $\Phi_\alpha(x_j)$, $\tilde{\Phi}_\gamma(x_j)$ are nodal basis functions, while $V_{\alpha i}(x_j)$, $P_\gamma(x_j)$ denote nodal values of the velocity and pressure to be computed. Substituting this approximation into the variational equations (7) and (8), the standard semidiscrete deterministic system can be conveniently expressed as (Bathe, 1996; Carey and Oden, 1986):

$$\mathbf{M}_V\dot{\mathbf{V}} + (\mathbf{K}_{\mu VV} + \mathbf{K}_{VV})\mathbf{V} + \mathbf{B}_{VP}\mathbf{P} = \mathbf{R}_B + \mathbf{R}_S, \tag{37}$$

$$\mathbf{B}_{VP}^T\mathbf{V} = 0, \tag{38}$$

where $\mathbf{M}_V$ is the mass matrix, $\mathbf{K}_{\mu VV}$ is the contribution from the viscous term, $\mathbf{K}_{VV}$ is the convective term and $\mathbf{B}$, $\mathbf{B}^T$ correspond to the discrete gradient and divergence operators and $\mathbf{R}_B$ and $\mathbf{R}_S$ correspond to body force and boundary data contributions. For example, the 2D flow case may be written in matrix notation as

$$\begin{bmatrix} \mathbf{M}_{V_k} & 0 & 0 \\ & \mathbf{M}_{V_l} & 0 \\ \text{symm.} & & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{V}}_k \\ \dot{\mathbf{V}}_l \\ \dot{\mathbf{P}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{\mu V_k V_k} + \mathbf{K}_{VV_k} & \mathbf{K}_{\mu V_k V_l} & \mathbf{B}_{V_k P} \\ \mathbf{K}_{\mu V_k V_l} & \mathbf{K}_{\mu V_l V_l} + \mathbf{K}_{VV_l} & \mathbf{B}_{V_l P} \\ \mathbf{B}_{TV_k P}^T & \mathbf{B}_{TV_l P}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_k \\ \mathbf{V}_l \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{\mathrm{B}_k} + \mathbf{R}_{\mathrm{S}_k} \\ \mathbf{R}_{B_l} + \mathbf{R}_{S_l} \\ 0 \end{bmatrix} \tag{39}$$

where V_k and V_l are the respectively nodal vectors of horizontal and vertical velocity components. The above system is assembled in the standard manner from element matrix and vector contributions (Carey and Oden, 1986). Let Φ^{e} and $\tilde{\Phi}^{\mathrm{e}}$ denote row vectors containing the corresponding velocity and pressure bases for element e. Then the element contributions to the global matrices in equation (39) can be conveniently expressed in the form:

$$\mathbf{M}^{\mathrm{e}}_{V_k} = \mathbf{M}^{\mathrm{e}}_{V_l} = \rho \int_{\Omega_{\mathrm{e}}} \Phi^{\mathrm{e}^{\mathrm{T}}} \Phi^{\mathrm{e}} \mathrm{d}x, \tag{40}$$

$$\mathbf{K}^{\mathrm{e}}_{\mu V_k V_k} = \int_{\Omega_{\mathrm{e}}} \left(2\mu \Phi^{\mathrm{e}^T}_{,k} \Phi^{\mathrm{e}}_{,k} + \mu \Phi^{\mathrm{e}^T}_{,l} \Phi^{\mathrm{e}}_{,l} \right) \mathrm{d}x, \mathbf{K}^{\mathrm{e}}_{\mu V_k V_l} = \int_{\Omega_{\mathrm{e}}} \mu \Phi^{\mathrm{e}^T}_{,l} \Phi^{\mathrm{e}}_{,l} \mathrm{d}x, \tag{41}$$

$$\mathbf{K}^{\mathrm{e}}_{\mu V_l V_l} = \int_{\Omega_{\mathrm{e}}} \left(2\mu \Phi^{\mathrm{e}^T}_{,l} \Phi^{\mathrm{e}}_{,l} + \mu \Phi^{\mathrm{e}^T}_{,k} \Phi^{\mathrm{e}}_{,k} \right) \mathrm{d}x, \tag{42}$$

$$\mathbf{K}^{\mathrm{e}}_{VV_k} = \mathbf{K}^{\mathrm{e}}_{VV_l} = \rho \int_{\Omega_{\mathrm{e}}} \left(\Phi^{\mathrm{e}^T} \Phi^{\mathrm{e}} \mathbf{V}_k \Phi^{\mathrm{e}}_{,k} + \Phi^{\mathrm{e}^T} \Phi^{\mathrm{e}} \mathbf{V}_l \Phi^{\mathrm{e}}_{,l} \right) \mathrm{d}x, \tag{43}$$

$$\mathbf{K}^{\mathrm{e}}_{V_k P} = - \int_{\Omega_{\mathrm{e}}} \Phi^{\mathrm{e}^T}_{,k} \tilde{\Phi}^{\mathrm{e}} \mathrm{d}x, \quad \mathbf{K}^{\mathrm{e}}_{V_l P} = - \int_{\Omega_{\mathrm{e}}} \Phi^{\mathrm{e}^T}_{,l} \tilde{\Phi}^{\mathrm{e}} \mathrm{d}x, \tag{44}$$

$$\mathbf{R}^{\mathrm{e}}_{B} = \int_{\Omega_{\mathrm{e}}} \Phi^{\mathrm{e}^T} \mathbf{f}_B \mathrm{d}x, \quad \mathbf{R}^{\mathrm{e}}_{S} = \int_{\partial\Omega_{\mathrm{e}}} \Phi^{\mathrm{e}^{ST}} \mathbf{f}_S \mathrm{d}(\partial x), \tag{45}$$

where components of the vector $\mathbf{f}_S$ are defined as

$$f_n = -p + 2\mu \frac{\partial v_n}{\partial n}, \quad f_t = \mu \left(\frac{\partial v_t}{\partial n} + \frac{\partial v_n}{\partial t} \right) \tag{46}$$

with v_n, v_t the normal and tangential boundary velocity components.

3.2 Stochastic finite element discretization

For convenience, let us assume that all input random fields are appropriately discretized relative to the associated finite element approximation problem, e.g. for the boundary fields in the later numerical test studies of Section 4 it suffices to use piecewise constant approximation of the random fields with values at the mid-edge points of elements on the boundary (with total number $\tilde{E}$). Therefore, each new input random vector also has $\tilde{E}$ components, and the cross-correlation matrix given by equation (10) is symmetric, positive definite of rank $\tilde{E}$. Furthermore, all partial derivatives of the state parameters with respect to input random variables are also determined at the edge mid points.

Following the approach in subsection 2.2, let us represent the random field vector b_r by the global vector of element random variables β_ξ, $\xi = 1, 2, \ldots, \tilde{N}$, $\tilde{N} = R \times \tilde{E}$, where R is the number of different input random fields. Next, as in equations (12) and (13), a perturbation expansion is employed for the relevant functions and parameters, and these expressions are now inserted in the corresponding finite element variational problems that lead to the expected values $E[\beta_\xi] \equiv \beta_\xi^0$, as in equation (26). That is, for a variable v

$$v = v^0 + \psi v^{,\xi}\Delta\beta_\xi + \frac{1}{2}\psi^2 v^{,\xi\zeta}\Delta\beta_\xi\Delta\beta_\zeta + \cdots + \frac{1}{n!}\psi^n v^{,n}(\Delta\beta)^n, \tag{47}$$

where $\psi\Delta\beta_\xi$, $\psi^2\Delta\beta_\xi\Delta\beta_\zeta, \ldots$ are defined as in equations (14)-(16).

Then, the stochastic finite element equations governing the fluid flow are (equations (19)-(24)): for the 0th order (ψ^0) terms, one coupled semidiscrete system for $P_\gamma^0(t)$ and $V_{i\alpha}^0(t)$

$$\mathbf{M}_V^0\dot{\mathbf{V}}^0 + \left(\mathbf{K}_{\mu VV}^0 + \mathbf{K}_{VV}^0\right)\mathbf{V}^0 + \mathbf{K}_{VP}^0\mathbf{P}^0 = \mathbf{R}_B^0 + \mathbf{R}_S^0, \tag{48}$$

$$\mathbf{K}_{TVP}^0\mathbf{V}^0 = 0, \tag{49}$$

corresponding to the deterministic model in equations (37) and (38) for the 1st order (ψ^1) terms, R systems, $r = 1, \ldots, R$

$$\begin{aligned} \mathbf{M}_V^0\dot{\mathbf{V}}^{,r} + \left(\mathbf{K}_{\mu VV}^0 + \mathbf{K}_{VV}^0\right)\mathbf{V}^{,r} + \mathbf{K}_{VP}^0\mathbf{P}^{,r} = \mathbf{R}_{\mathrm{B}}^{,r} + \mathbf{R}_{\mathrm{S}}^{,r} \\ - \mathbf{M}_V^{,r}\dot{\mathbf{V}}^0 + \left(\mathbf{K}_{\mu VV}^{,r} + \mathbf{K}_{VV}^{,r}\right)\mathbf{V}^0 + \mathbf{K}_{VP}^{,r}\mathbf{P}^0, \end{aligned} \tag{50}$$

$$\mathbf{K}_{TVP}^0\mathbf{V}^{,r} = -\mathbf{K}_{TVP}^{,r}\mathbf{V}^0, \tag{51}$$

for the 2nd order (ψ^2) terms, one system of the form

$$\begin{aligned} \mathbf{M}_V^0\dot{\mathbf{V}}^{,rs} + \left(\mathbf{K}_{\mu VV}^0 + \mathbf{K}_{VV}^0\right)\mathbf{V}^{,rs} + \mathbf{K}_{VP}^0\mathbf{P}^{,rs} = \mathbf{R}_B^{,rs} + \mathbf{R}_S^{,rs} \\ - \left\{\mathbf{M}_V^{,rs}\dot{\mathbf{V}}^0 + 2\mathbf{M}_V^{,r}\dot{\mathbf{V}}^{,s} + \left(\mathbf{K}_{\mu VV}^{,rs} + \mathbf{K}_{VV}^{,rs}\right)\mathbf{V}^0\right\} \\ - \left\{2\left(\mathbf{K}_{\mu VV}^{,r} + \mathbf{K}_{VV}^{,r}\right)\mathbf{V}^{,s} + \mathbf{K}_{VP}^{,rs}\mathbf{P}^0 + 2\mathbf{K}_{VP}^{,r}\mathbf{P}^{,s}\right\} \end{aligned} \tag{52}$$

$$\mathbf{K}_{TVP}^0\mathbf{V}^{,rs} = -\left(\mathbf{K}_{TVP}^{,rs}\mathbf{V}^0 + 2\mathbf{K}_{TVP}^{,r}\mathbf{V}^{,s}\right). \tag{53}$$

More generally, we have for the n-th order system

$$\begin{aligned} \sum_{k=0}^{n}\binom{n}{k}\mathbf{M}_V^{(k)}\dot{\mathbf{V}}^{(n-k)} + \sum_{k=0}^{n}\binom{n}{k}\left(\mathbf{K}_{\mu VV}^{(k)} + \mathbf{K}_{VV}^{(k)}\right)\mathbf{V}^{(n-k)} \\ + \sum_{k=0}^{n}\binom{n}{k}\mathbf{K}_{VP}^{(k)}\mathbf{P}^{(n-k)} = \mathbf{R}_B^{(n)} + \mathbf{R}_S^{(n)}, \end{aligned} \tag{54}$$

$$\sum_{k=0}^{n} \binom{n}{k} \mathbf{K}_{TVP}^{(k)} \mathbf{V}^{(n-k)} = 0, \tag{55}$$

Remark. Note that in the classical regular perturbation approaches the coefficient matrices in the above case may be fixed. It follows that in this situation a single factorization may be used with successive substitution solves as the right hand sides change.

Integrating the successive perturbation equations sequentially (or treating sequentially within a time step), the expected values of the fluid pressures and velocities at any time $t = \tau$ can be evaluated for the 2nd order approximation as (equation (26))

$$E[f_{\alpha}(\tau)] = f_{\alpha}^{0}(\tau) + \frac{1}{2} f_{\alpha}^{(2)}(\tau) \tag{56}$$

and the corresponding space-time cross-covariance is given by (equation (31))

$$\mathrm{Cov}(f_{\alpha}(t_1), f_{\beta}(t_2)) = f_{\alpha}^{,\xi}(t_1) f_{\beta}^{,\zeta}(t_2) S_{b}^{\xi\zeta}. \tag{57}$$

The above formulations complete the description of an n-th order perturbation second probabilistic moment finite element analysis for viscous incompressible flow. A similar analysis and formulation can be applied for divergence free spaces and in this case the pressure terms are not explicitly present. Other simplifications of the system and resulting formulation follow by simply considering the reduced forms of the equations. For example, we illustrate the approach for simple Couette flow with random input or potential flow with random boundary effects in the following section. This permits us to write down the SFEM contributions in a compact form and explicitly validate the approach for a case with known analytic solution, from which the analytic expectation and variance can be calculated for comparison.

4. Numerical studies

Channel and pipe flow with various random fields specified and potential flow with random variation for the body shape are considered as illustrative test and verification problems. Both analytical and finite element approximate formulations are computed. Note that this general approach requires computation of derivatives with respect to the random variable (equation (28)). This differentiation can be carried out at the code level or numerically by differencing, or using a symbolic manipulator such as in the MAPLE implementation employed here. The MAPLE implementation allows us to exploit not only this symbolic capability but also the statistical utility routine in MAPLE. This enables relatively easy extension of the technique to higher perturbation order, convergence analysis for particular probabilistic moments, integration with FORTRAN codes for FEM programs as well as efficient visualization of the results.

4.1 Channel flow problem

To demonstrate the main ideas and results in a simple possible setting, let us consider the problem for viscous incompressible flow between parallel plates with separation H. The flow equations simplify to a linear differential equation (Spurk, 1997; White, 1986) with the tangential fluid velocity at the upper plate being specified. The analytic solution to the deterministic problem with specified pressure gradient ($p_x \neq 0$) is

$$\rho\nu u = \frac{y^2}{2}\left(\frac{\partial p}{\partial x}\right) + \frac{y}{H}\left\{\rho\nu U - \frac{H^2}{2}\left(\frac{\partial p}{\partial x}\right)\right\}, \tag{58}$$

where $\nu = \mu/\rho$ denotes the kinematic viscosity (Wang *et al.*, 1990; Zaradny, 1993). In the present study we are interested in the effect of uncertainty in the lid velocity U, the viscosity, or the channel width H. These respective situations are now considered.

4.1.1 Couette flow validation: random velocity U. Let the tangential velocity U of the upper plate be the input Gaussian random variable given by its first two probabilistic moments $E[U]$, Var(U), respectively. The rest of the constitutive and geometrical parameters are treated as deterministic. Then, for the simplest case of zero pressure gradient, the SFEM systems for the 0th, 1st and 2nd order terms reduce to

$$\mathbf{K}^0\mathbf{u}^0 = 0, \quad \mathbf{K}^0\mathbf{u}^{,r} = -\mathbf{K}^{,r}\mathbf{u}^0, \quad \mathbf{K}^0\mathbf{u}^{(2)} = -(\mathbf{K}^{,rs}\mathbf{u}^0 + 2\mathbf{K}^{,r}\mathbf{u}^{,s})S^{rs}. \tag{59}$$

For this simple validation case we use two linear finite elements between the plates with nodes 1, 2, 3 at $y = 0$, $y = H/2$ and $y = H$, respectively, on an arbitrary section. Then equation (59) simplified to

$$\left(\frac{2\mu}{H}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\right)^0 \begin{Bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{60}$$

$$\left(\frac{2\mu}{H}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\right)^0 \begin{Bmatrix} u_1^{,U} \\ u_2^{,U} \\ u_3^{,U} \end{Bmatrix} = -\left(\frac{2\mu}{H}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\right)^{,U} \begin{Bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{Bmatrix}, \tag{61}$$

$$\left(\frac{2\mu}{H}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\right)^0 \begin{Bmatrix} u_1^{,UU} \\ u_2^{,UU} \\ u_3^{,UU} \end{Bmatrix} = -\left(\frac{2\mu}{H}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\right)^{,UU} \begin{Bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{Bmatrix} - \left(\frac{4\mu}{H}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\right)^{,U} \begin{Bmatrix} u_1^{,U} \\ u_2^{,U} \\ u_3^{,U} \end{Bmatrix}. \tag{62}$$

Using equations (54) and (55) together with the relation for the variance (equation (35)), higher-order contributions can be determined similarly.

The expectation and variance for centerline nodal velocity u_2 with 2nd order model follow from the solution of equations (60)-(62) as

$$E[u_2] = u_2^0 + \frac{1}{2} u_2^{;UU} \mathrm{Var}(U) = \frac{1}{2} E[U] \tag{63}$$

and

$$\mathrm{Var}(u_2) = u_2^{;U} u_2^{;U} \mathrm{Var}(U) = \frac{1}{4} \mathrm{Var}(U). \tag{64}$$

This result can be compared with probabilistic moments derived explicitly from the analytic solution $u = (y/H)U$ and we have

$$E[u|_{y=H/2}] = \frac{E[U]}{2}, \quad \mathrm{Var}(\mathrm{u}|_{y=H/2}) = \frac{\mathrm{Var}(U)}{4}, \tag{65}$$

which is identical to our SFEM result in equations (63) and (64).

Further, the shear stresses acting on the moving plate are $\sigma_{yx} = (\mu/H)U$, which, for randomly defined velocity of the upper plate as given by its first two probabilistic moments $E[U]$ and Var(U), implies

$$E[\sigma_{yx}] = \frac{\mu}{H} E[U], \quad \mathrm{Var}(\sigma_{yx}) = \frac{\mu^2}{H^2} \mathrm{Var}(U). \tag{66}$$

The second moment perturbation approach and finite element model of equations (60)-(62) lead to the following result:

$$E[\sigma_{yx}] = \sigma_{yx}^0 + \frac{1}{2} \sigma_{yx}^{;UU} \mathrm{Var}(U) = \frac{\mu}{H} E[U] \tag{67}$$

$$\mathrm{Var}(\sigma_{yx}) = \left(\sigma_{yx}^{;U}\right)^2 \mathrm{Var}(U) = \frac{\mu^2}{H^2} \mathrm{Var}(U), \tag{68}$$

which again agree with the theoretical result (66). This exact agreement results from the fact that the probabilistic output random field is a linear function of the input random parameter and the exact solution to the deterministic problem is in the finite element state.

4.1.2 Viscosity uncertainty. The situation is more complicated in the case where the random variation is in the viscosity. The symbolic MAPLE code is applied as before but now with $H = 2.0$, $E[\mu] = 1.0$, $U = 10.0$.

Since a Gaussian input is considered once again, all central probabilistic moments can be recovered from its expected value and coefficient of variation (being a design variable in this study) using the formulas (29) and (30). This problem is a good test case of probabilistic convergence because, for the n-th order approach, it consists of n non-trivial algebraic systems

$$\mathbf{K}^0\mathbf{u}^0 = 0,\ \mathbf{K}^0\mathbf{u}^{,\mu} = -\mathbf{K}^{,\mu}\mathbf{u}^0,\ K^0 u^{,\mu\mu} = -2K^{,\mu} u^{,\mu}, \ldots, \mathbf{K}^0 \frac{\partial^n \mathbf{u}}{\partial \mu^n} = -n\mathbf{K}^{,\mu} \frac{\partial^{n-1}\mathbf{u}}{\partial \mu^{n-1}}, \tag{69}$$

and, therefore, each new perturbation order introduces new components into the probabilistic output. This convergence study with increasing order is an important aspect of the present work. Recall also that any higher order derivative of the system matrix with respect to fluid viscosity is equal to 0. For each case we graph the fluid velocity along the horizontal top channel section as a function of two independent variables – the perturbation order and the coefficient of variation of the random input. Note that it is also possible to compute and to analyze probabilistic moments of higher order. Once again the symbolic capability of the MAPLE implementation facilitates such computations. The expected values of the maximum fluid velocity at $y=H/2$ for 2nd, 4th, 6th, 8th and 10th order stochastic finite element approximation are collected in Figure 1 as functions of the perturbation parameter ψ (left horizontal axis – *psi*) in the interval (0.8, 1.2) and the input coefficient of variation for the fluid viscosity ("alfami" – horizontal axis at right), from 0.0 (as for deterministic test) to about 0.3. Results for standard deviations are shown in Figure 2, while the output coefficient of variance is shown in Figure 3. It is clear from these graphs that the convergence of the SFEM strongly depends on the input coefficient of variation. For smaller values like 0.1 the 2nd order method may be sufficient, but the maximum value of 0.3 needs at least a 10th order approximation for similar accuracy.

The use of symbolic calculus also allows us to recover polynomial expressions for any moment (which is not possible in standard FEM computations). Using the SFEM approach, polynomial expressions for the expected values and standard deviations in

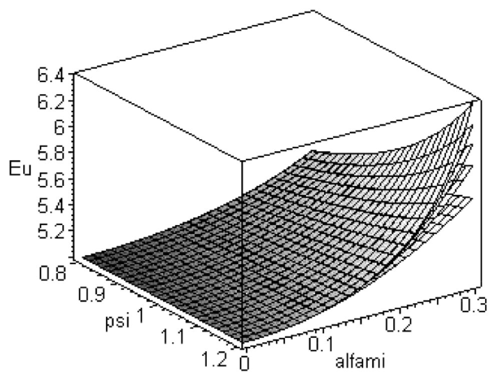


Figure 1.
Expected values of fluid velocity for random viscosity, 2, 4, 6, 8, 10th orders

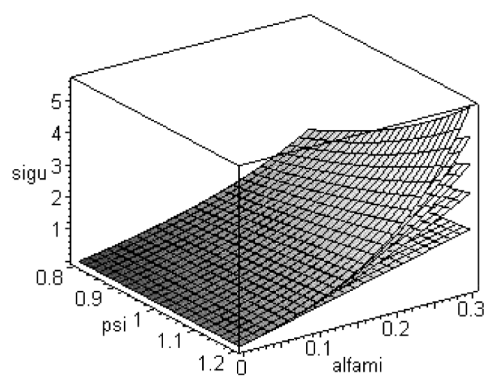


Figure 2.
Standard deviations of fluid velocity for random viscosity, 2, 4, 6, 8, 10th orders

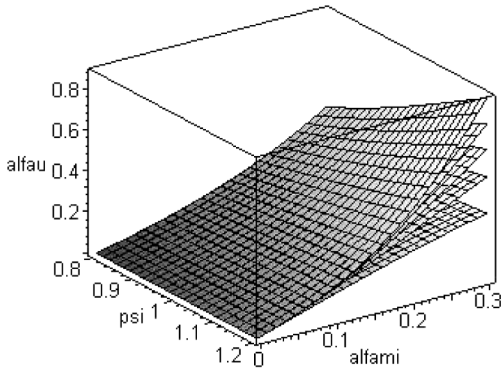


Figure 3.
Coefficients of variation of fluid velocity for random viscosity, 2, 4, 6, 8, 10th orders

the 10th order approximation are returned symbolically by MAPLE. For example, in the present case we obtain

$$E[u(\psi, \alpha(\mu))] = 0.25\frac{UH}{E[\mu]} + 5.0\psi^2\alpha^2(\mu) + 15.0\psi^4\alpha^4(\mu) + 75.0\psi^6\alpha^6(\mu) + 525.0\psi^8\alpha^8(\mu) + 4725.0\psi^{10}\alpha^{10}(\mu) \quad (70)$$

and

$$\sigma(u(\psi, \alpha(\mu))) = 10\psi\alpha(\mu) + 8.6602\psi^2\alpha^2(\mu) + 19.3649\psi^3\alpha^3(\mu) + 51.2348\psi^4\alpha^4(\mu) + 153.7043\psi^5\alpha^5(\mu) \quad (71)$$

The expected values, standard deviations and the coefficients of variation are shown with respect to perturbation parameter only in Figures 4-9. Here they are computed for $\alpha(\mu) = 0.25$ (out of validity range for the 2nd order technique – Figures 4-8) and $\alpha(\mu) = 0.10$ (acceptable for the 2nd SFEM, cf. Figures 5-9).

It is evident from these results that the coefficient of variation of the input random variable is again more influential than the perturbation parameter in so far as the probabilistic moments and characteristics of the solution vector component are concerned. The extended 10th order perturbation technique displayed here appears to

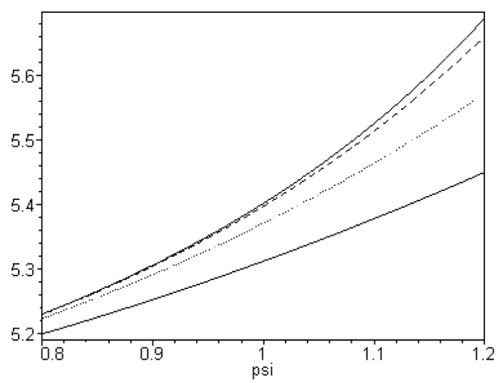


Figure 4.
Expected values of fluid velocity for random viscosity, 2, 4, 6, 8, 10th orders, $\alpha(H) = 0.25$

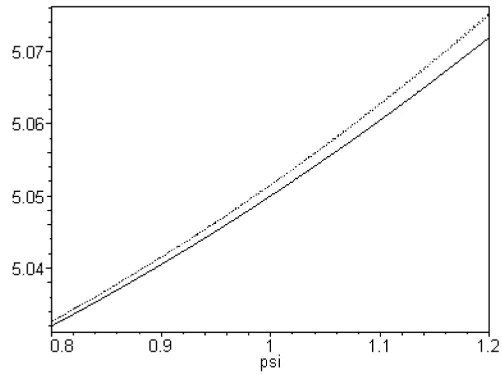


Figure 5.
Expected values of fluid velocity for random viscosity, 2, 4, 6, 8, 10th orders, $\alpha(H) = 0.10$

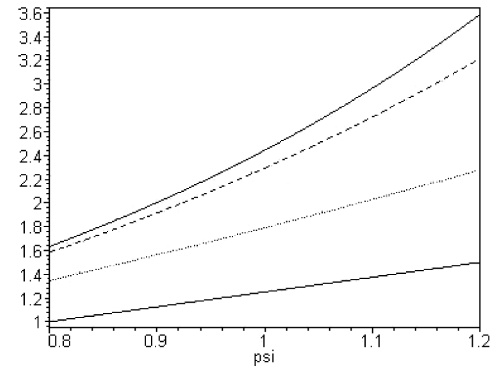


Figure 6.
Standard deviations of fluid velocity for random viscosity, 2, 4, 6, 8, 10th orders, $\alpha(H) = 0.25$

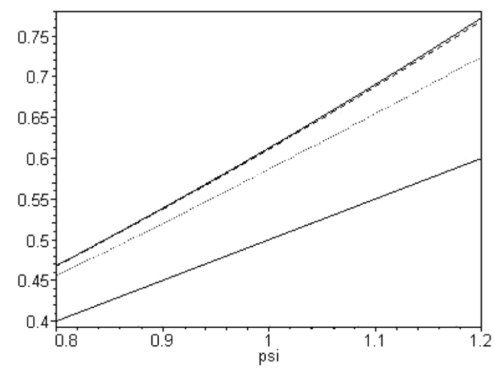


Figure 7.
Standard deviations of fluid velocity for random viscosity, 2, 4, 6, 8, 10th orders, $\alpha(H) = 0.10$

be effective even with 30 percent dispersion of the random input. However, this level of dispersion is quite unacceptable for the accuracy of the previous 2nd order SFEM. As shown in Figures 5 and 7, the expected values and standard deviations are almost insensitive to the perturbation parameter for smaller coefficients of variation. The significance of this parameter increases with both order of perturbation method and the coefficient of variation for random input.

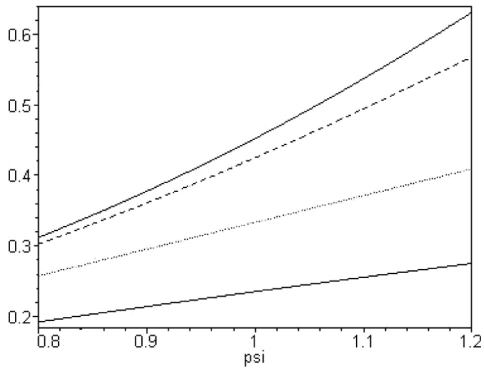


Figure 8.
Coefficients of variation of fluid velocity for random viscosity, 2, 4, 6, 8, 10th orders, $\alpha(H) = 0.25$

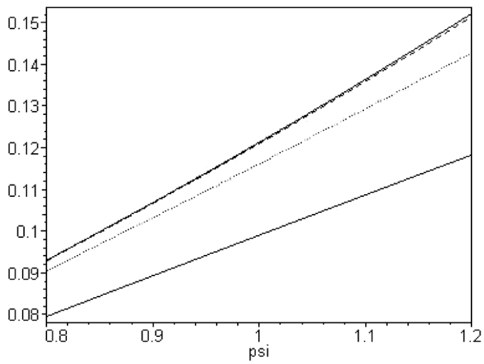
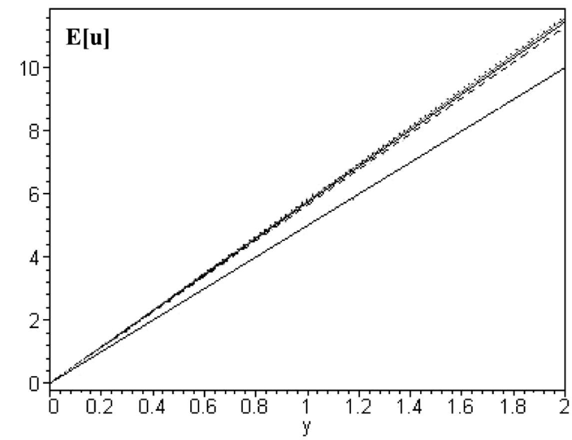


Figure 9.
Coefficients of variation of fluid velocity for random viscosity, 2, 4, 6, 8, 10th orders, $\alpha(H) = 0.10$

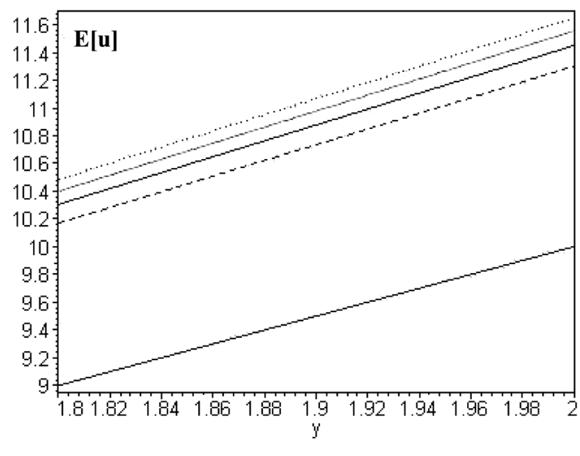
Further comparing Figure 5 with 7, it is clear that probabilistic convergence for the expected values is significantly faster than observed for the standard deviations. As one might infer from asymptotic theory, the influence of the perturbation parameter increases with the order of the method – the higher the order of the generalized SFEM, the faster the increase of the results corresponding to greater $\alpha(\mu)$. These differences are more apparent in Figure 8. Comparison of Figures 8 and 9 shows that for $\alpha(\mu) = 0.10$ the 2nd order technique is quite satisfactory, while $\alpha(\mu) = 0.25$ needs an 8th or 10th order expansion for the same tolerance. These differences are more significant for the 2nd order characteristics, where even for $\alpha(\mu) = 0.10$ at least a 4th order perturbation technique would be necessary. Figures 7 and 9 show that standard deviations and coefficients of variation with $\psi = 1$ computed for 2nd order are about two times smaller than the result obtained for a 10th order expansion. It should also be remarked here that the convergence of the stochastic perturbation-based Finite Element Method may be corrected for smaller orders by the appropriate choice of perturbation parameter ψ, which needs to be greater than 1 in this case.

4.1.3 Channel width H *fluctuations*. The analytic solution here is complicated by the fact that the channel height enters in the denominator. Therefore, for any new random variable $Y = 1/H$ its expected value does not exist in the general case and may be evaluated analytically for finite intervals only. However, in the perturbation approach this difficulty does not arise and instead we can evaluate the expectation directly by a formula (25) or (26). Deterministic values are compared here against the 2nd, 3rd, 4th, 5th and 6th order perturbation based approximation of the expected values for $U = 10.0$, $E[H] = 2.0$ and its coefficient of variation equal to 0.1. The perturbation parameter is again taken as 1. The expected values are collected in Figure 10(a) for the entire channel height and in Figure 10(b) near the moving plate. This example with H being random is an interesting test case because there are an infinite number of nonzero partial derivatives with respect to the random input parameter and hence each new perturbation order introduces new components into the probabilistic output.

In the case of a linear or even a polynomial transformation between output and input, the differences between the results obtained for various perturbation-based approximations essentially decrease. Furthermore, as expected, the accuracy improves as the perturbation parameter decreases. Generally, we observe that application of



(a)



(b)

Figure 10.
(a) Stochastic convergence of the perturbation method for random channel height; and (b) stochastic convergence of the perturbation method for random channel height

a polynomial transformation can be successful even in the case of a 2nd order approximation with perturbation parameter equal to 1, whereas in the case of inversion of the random input (*H*), the convergence is very slow and that is why reducing the coefficient ψ is recommended.

The simple Couette flow form is again considered using the stochastic finite element approach. This implies that cross-correlations between various elements are equal to the product of standard deviations of finite elements lengths. The computations are performed for various coefficients of variations of these lengths varying from 0.05 to 0.15 in increment of 0.01. The results of these finite element computations are shown in Figure 11 for a discretization that has 20 linear finite elements across the channel. Computational analysis here is performed using a perturbation-based SFEM program called RANDFLOW, which is implemented from equation (59) to calculate 0th, 1st and 2nd order fluid velocities and subsequently the relevant expected values and variances. This program is implemented in such a way that the input random field components are defined at the finite element midpoints, where the expected values and cross-correlation matrix components are given as the input. Boundary conditions are introduced in the following manner: fluid velocity is equal zero at the lower boundary and the surface traction is $\hat{f} = 0.5$ at the upper edge.

The expected values show linear variability with respect to the fluid flow profile. Comparing the results for various input coefficients of variation it is seen that parameter variation results in very small changes of the fluid velocity expectations examined. Essential changes are observed in the case of variances that have parabolic variability with respect to the varying vertical coordinate. The differences between the profiles computed for increasing input coefficients of variation also systematically grow. Further, the system demonstrates a linear behavior in the sense that the input coefficient of variation is exactly equal to the output one, as seen from the results collected in Figure 11.

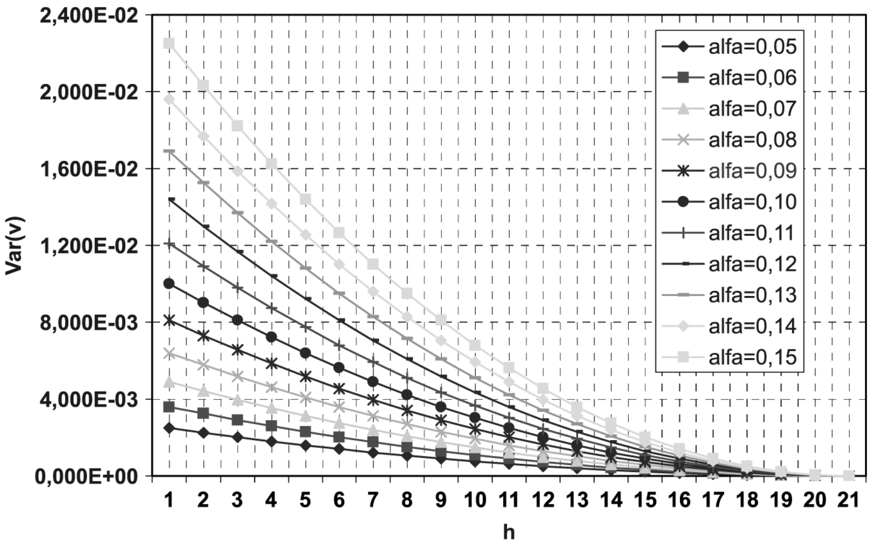


Figure 11.
Variance of velocities in stochastic 1D flow

4.2 Pipe with random diameter – validation test

The behavior in pipe flow is analogous to that of the preceding channel flow problem but is of more general engineering interest. The pipe has the viscous fluid flow through the pipe with constant diameter D, and constant pressure gradient G, so the velocity profile is described by the following relation:

$$u = \frac{G}{4\mu}\left(\frac{D^2}{4} - r^2\right). \tag{72}$$

The pipe diameter D is taken as input Gaussian random variable given by the first two moments $E[D]$ and $\mathrm{Var}(D)$. In this particular case of a transformation of the 2nd order Gaussian random variable we can derive an exact solution using the following formulas:

$$E[D^2] = E^2[D] + \mathrm{Var}(D), \quad \mathrm{Var}(D^2) = \mathrm{Var}(D)(2E^2[D] + \mathrm{Var}(D)). \tag{73}$$

Then, an exact solution is obtained for randomized diameter in equation (72) as

$$E[u(y)] = \frac{G}{4\mu}\left(\frac{E^2[D]}{4} - r^2(y)\right), \quad \mathrm{Var}(u) = \left(\frac{G}{16\grave{\imath}}\right)^2 \mathrm{Var}(D)(2E^2[D] + \mathrm{Var}(D)). \tag{74}$$

Therefore, a comparison with MCS (treated as numerically exact solution) is not necessary and terms higher than a 2nd order in the perturbation expansion are equal to zero. Let us note, moreover, the 2nd order moments are constant along the cross-section of a pipe. Furthermore, comparison of expected values in equation (74) with (72) shows that the deterministic value is negligibly smaller than the exact solution for the expected value and is omitted in numerical comparison as far as deterministic parameter D is equal to its mean value in statistical measurements. Perturbation based computational studies are performed using the following data: $G = 10$, $\mu = 20$, $D = E[D] = 0.5$. The deterministic velocity profile is compared against the perturbation-based expected value for spatial distribution in Figures 12 and 13 below. The 2nd order perturbation result in Figure 12 is greater than the deterministic result and exact solution for the probabilistic problem. This follows from the fact that in the perturbation solution the deterministic value is corrected with the 2nd order term, which being positive increases the final result. The modeling error as measured by the difference between the perturbation-based expected value and the deterministic

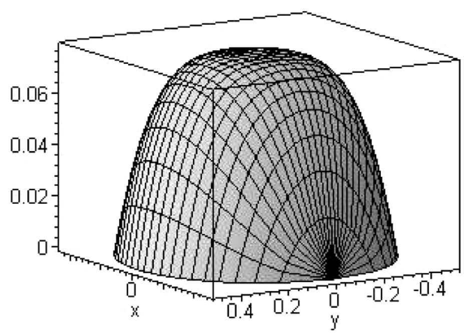


Figure 12.
Deterministic profile of the flow through pipe with constant diameter

value is constant in the pipe cross-section and is obtained from the MAPLE code as 17.5781×10^{-4} for these data.

The following graph (Figure 14) contains the standard deviations of the velocity profile for the exact probabilistic solution (dashed line) and for the perturbation-based formula (solid line). Since the standard deviation of the profile has constant value independent of r, its variability is considered with respect to input coefficient of variation of the pipe diameter. As seen in Figure 14, the perturbation solution returns larger values than the exact solution for any choice of the input coefficient of variation of random variable D; in both cases the increase of the output standard deviation is proportional to that of the input parameter results. For input coefficient of variation equal to 0.10 being an upper bound for the 2nd order applicability, the modeling error defined as before as the difference between perturbation and exact solutions, is equal here to 4.5487×10^{-2}.

4.3 Potential flow past a cylinder – random radius fluctuations

The next numerical example is for potential flow past a cylinder with random radius fluctuations and is also solved using the symbolic manipulation capability of mathematical package MAPLE (Cornil and Testud, 2001). Recall the related classical wavy wall perturbation problem referred to in the Introduction (Lighthill, 1954).

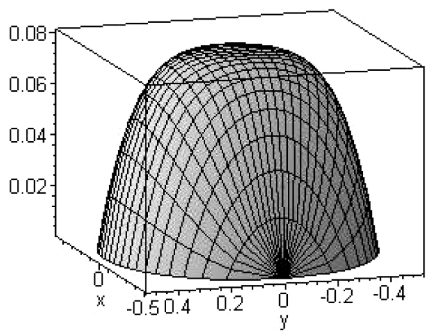


Figure 13.
Expected value of a profile for the flow through pipe with constant diameter

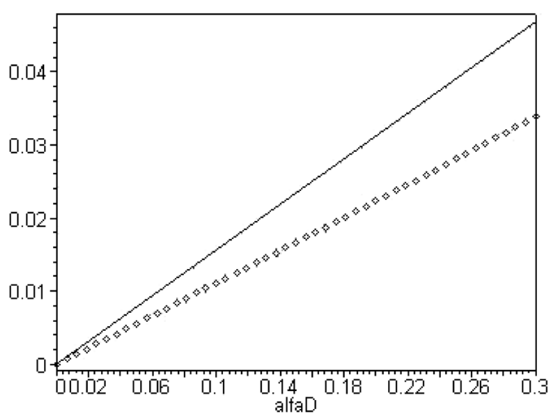


Figure 14.
Standard deviations of a profile for the flow through pipe

The analytical (deterministic) solution for the velocity potential exterior to a cylinder follows from elementary complex variable theory and is

$$\varphi = U\left(r + \frac{a^2}{r}\right)\cos\theta + 0.50, \tag{75}$$

where U is the remote uniform free stream velocity and a is the radius of the cylinder. In the present work, the original MAPLE script has been extended to include the 2nd order second moment equation for the probabilistic moments of the input. The Gaussian input random variable in the computations is the cylinder radius and is defined using its expected value $E[a]$ and variance Var(a). The stochastic 2nd order perturbation algorithm returns the expected values and the variances of the velocity potential as

$$E[\varphi] = \varphi^0 + \frac{1}{2}\varphi^{(2)} = U\left(r + \frac{E^2[a]}{r}\right)\cos\theta + 0.50 + U\frac{1}{r}\cos\theta\,\mathrm{Var}(a), \tag{76}$$

$$\mathrm{Var}(\varphi) = \left(\frac{\partial\varphi}{\partial a}\right)^2 \mathrm{Var}(a) = 4U^2\frac{a^2}{r^2}\cos^2\theta\,\mathrm{Var}(a). \tag{77}$$

These results are now used to compare deterministic streamlines, their expected values, 1st and 2nd order partial derivatives with respect to random input variable as well as variances for specific values of uniform velocity U. The expected value of the cylinder radius is $E[a] = 1$ and its standard deviation is taken to be 10 percent of the expectation. The main results are presented graphically as streamline patterns and their variations in Figure 15 below using the MAPLE visualization tool.

It was verified numerically that deterministic and expected values of the streamlines obtained for a flow with various speeds are almost identical since the contributions of the 2nd order terms to the expectation are very small. As presented next, the 1st and 2nd order partial derivatives of the streamline functions differ from zero in this particular problem (Figure 15(b)). Since partial derivatives of third and higher order are equal to zero, computational implementation of higher than the 2nd order perturbation approach is not necessary in this case. Moreover 1st and 2nd order components differ here by a constant, so the streamlines for both components are exactly the same. Because the 2nd order derivatives are nonzero, we can determine the variances of streamlines, which are computed symbolically for higher flow speeds only (Figure 15(c)). (In the case of smaller values of the parameter U these variations are close to zero and, therefore, the automatic resolution of the MAPLE graphic procedures is too small for reasonable visualization.) In both cases these streamlines show double – horizontal and vertical symmetry with respect to the axis crossing the cylinder geometrical center. The streamline variances cover an increasing area of the flow domain for increasing inlet velocity.

5. Concluding remarks

This paper describes a formulation of the perturbation stochastic second moment approach for analysis of fluid flow involving random field parameters. Elementary computational tests for potential flow past a circular cylinder and for the viscous

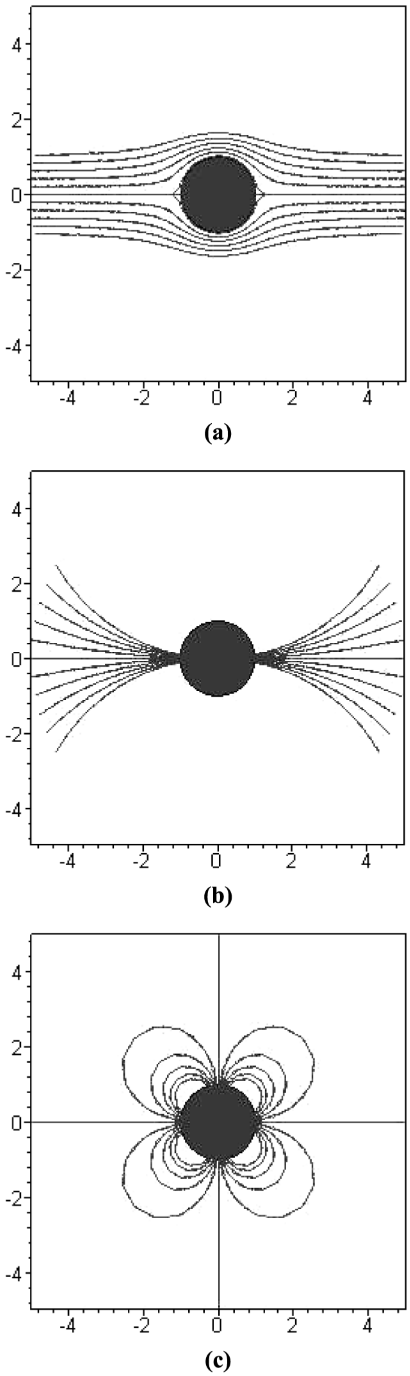


Figure 15.
(a) Expectations of streamlines for $U = 2.0$; (b) 1st and 2nd order streamlines for $U = 10.0$; and (c) variance of streamlines for $U = 10$

Couette flow problem with random fluctuations in boundary shape and boundary conditions are solved to demonstrate the basic ideas and validate against simple analytic results. As seen from the numerical experiments, implementation of the stochastic perturbation technique for fluid flow introduces some issues in the SFEM implementation and further numerical studies are needed to compare probabilistic moments computed with other stochastic methods. For instance, improved Monte-Carlo analyses of the same flow problems are desirable to determine the most efficient perturbation orders for various input and perturbation parameters. This can be performed using, e.g. the probabilistic module of the commercial FEM package ANSYS, v. 5.7 or by extension of existing academic flow solvers with computational random generators and statistical estimators.

In the finite element formulation, the first two probabilistic space and time characteristics of the problem state functions (e.g. fluid pressure and velocities) are first computed and the expected values and covariances of the state parameters are then determined. The equations (66)-(68) are also useful in SFEM-based reliability studies of fluid-structure interaction problems. In this case, a reliability index can be derived in a manner similar to that presented in Ghanem and Spanos (1991) using the first order reliability method (FORM), the second order reliability method (SORM) or in Kamiński (2001b) for the Weibull second order third moment (W-SOTM) perturbation technique. Let us note that probabilistic approaches higher than 2nd order for non-Gaussian random variables should be applied for all those cases, where skewness of the input random parameters cannot be neglected.

Computational analysis of stochastic sensitivity (Kleiber and Hien, 1992) for the fluid flow problem can be carried out with respect to physical properties of the fluid as well as geometrical parameters of the flow. For example, the ideas presented in the paper may be useful for solution of flow problems involving random perturbations of fluid suspensions and foams (Carey and Hu, 2005; Carey *et al.*, 1996; Carey and Oden, 1986), where constitutive parameters of both fluid and/or multiphase medium are random fields (Kamiński, 2001a). In this case, the deterministic characteristic function is substituted into the derived equations. Random fluctuations due to boundary bubbles (Boutin and Auriault, 1993) appearing in real fluids, such as those due to temperature effects near a phase change boundary may be similarly described stochastically using the first two probabilistic moments with respect to their radii and the total number in the analyzed region. Finally, in view of recent progress with the SFEM models in solids (Grigoriu, 2000), the approach appears applicable to analysis of random uncertainties at interfaces in fluid-solid interaction problems. The probabilistic moments computed in the manner described here can be used in the stochastic analysis of reliability problems, where the limit function is defined using actual and allowable shear stresses induced by a moving fluid on the solid part. Probabilistic moments of fluid velocities and of pressures would be included in the final formula characterizing the reliability index.

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